

Fine-Tuning LLMs for Real Analysis Problems

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Abstract

Large language models usually struggle with complex mathematical questions. Previous research primarily focuses on incorporating supplementary tools or functionality to provide the language models additional layers of context needed to solve such queries. Building upon these existing ideas, we propose an LLM that is capable of providing relevant lemmas, definitions, and other antecedent theorems that correspond to the input mathematical question. Specifically, we fine-tune gpt-3.5-turbo on a corpus of Real Analysis texts, with the objective of providing helpful and pertinent information for Real Analysis questions found on the Harvard Math Entrance Exam.

1. Previous Work

Yiran Wu, Feiran Jia, et al. chained GPT-4 and Python to solve mathematical problems. They used GPT-4 to recognize math problems, then generate related python code. From GPT-4's response, the user proxy agent extracts all code and executes them sequentially. Valid code from previous runs is recorded and will be executed together with the new code to reflect the step-by-step reasoning progress of the model. The results will be returned to GPT-4 and GPT-4 will continue its problem-solving process. [1]

Imani, Du and Shrivastava used zero-shot chain-of-thought prompting technique to generate multiple Algebraic expressions or Python functions [2]. They achieved a high confidence level on their results and improved upon the then-golden standard MultiArith dataset. Wang and Hu used the same technique on more challenging questions and prompts. Their process looked like this:

Stage 1: Input: [input-question] Let's think step by step.

Output: [explanation]

Stage 2: Input: [input-question] Let's think step by step. [explanation] + Therefore, the answer is:

Output: [answer] [3]

2. Problem Statement and Goal

2.1. Problem Statement

State of the art large language models such as GPT-4 have demonstrated these architectures' superb ability at text completion for natural language prompts. However, a common pitfall of these models is their inherent predisposition to hallucinate information or reasoning when it comes to factual based queries. To date, there has been no meaningful progress in the open problem of getting LLMs to produce sound and accurate proofs to solve mathematical questions. Namely, these models are currently unable to pass the Harvard Math entrance exam, which demonstrates mastery of various higher level math fields such as Analysis, Topology, Algebra and more. While the training data for GPT-4 is not available to the general public, GPT-4 does demonstrate an 'understanding' of these concepts and common questions relating to them, suggesting that it has been exposed to the respective fields at a sufficient level in its training process.

Thus, despite having enough exposure to understand these questions, GPT-4 lacks the ability to close the gap between the prompt question and the derivation of the proof.

In this work, we first extract all theorems and definitions from a myriad of real analysis texts [4–8], and then use these to fine tune an instance of gpt-3.5-turbo, that we coin the Lemma Finder, in order to approximate the underlying logical transformation that would allow any theorems that imply the one in question to be derived. Once trained, we then feed the Real Analysis questions from the Harvard Math Entrance exam to this fine-tuned gpt-3.5-turbo model, and use the returned results as additional context in prompting GPT-4 for a solution.

While this pipeline was designed with the restriction to the Real Analysis field in mind, it may seamlessly be extended to other fields to provide for all encompassing model proof derivers.

2.2. Goal

To fine-tune gpt-3.5-turbo on (theorem, lemmas) pairs in Real Analysis texts to cultivate a model that is able to accurately provide relevant and supplemental theorems that could be used as a first non-trivial step in deriving solutions

for the real analysis questions from the Harvard Math Entrance Exams.

3. Dataset and Methods

3.1. Dataset

We curate our dataset using real analysis theorems sourced from mathematics textbooks [4–8]. The dataset will include theorem statements and their corresponding antecedent theorems used in their proofs, as well as definitions necessary for their constructions. Specifically, we cater our collection of textbooks to match the requirements for the Real Analysis portion of the qualifying exam syllabus of the Harvard Math Entrance Exam. [9].

3.1.1 Dataset Curation

In order to attain the dataset needed to fine tune the Lemma Finder, we had to create a structured representation of a theorem/definition/corollary/proposition label, its statement, proof and all the corresponding statements referenced in its proof.

To do so, we started by using the Mathpix API [10] to convert online PDF files of [4–6] into markdown files with the corresponding math and non-math text fully rendered. The code is available on GitHub, see [11].

While initially a simple regex pattern was considered, we realized early on that even within the scope of one author, the variation in how these mathematical statements were presented were too broad to be effectively captured by any classical, algorithmic approach.

1.37 Theorem Suppose $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^k$, and α is real. Then

- (a) $|\mathbf{x}| \geq 0$;
- (b) $|\mathbf{x}| = 0$ if and only if $\mathbf{x} = \mathbf{0}$;
- (c) $|\alpha\mathbf{x}| = |\alpha| |\mathbf{x}|$;
- (d) $|\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}| |\mathbf{y}|$;
- (e) $|\mathbf{x} + \mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|$;
- (f) $|\mathbf{x} - \mathbf{z}| \leq |\mathbf{x} - \mathbf{y}| + |\mathbf{y} - \mathbf{z}|$.

Figure 1. Theorem from [4] with multiple sub-statements.

Furthermore, in the cases where theorems would have multiple sub-statements (as seen in 1), that on their own would not give a complete definition, we would require an agent to interpolate the text so as to make any statement self-contained.

Our solution was to use a combination gpt-3.5-turbo-1106 and gpt-4-1106-preview with function calling to produce consistent JSON outputs with the desired text. Gpt-3.5-turbo-1106 was used exclusively for the cases where the task required the JSON mode feature of the openai models

alone. We found that with our detailed system prompt, it often produced incorrect JSON outputs or failed to properly inference the context needed for standalone statements. Gpt-4-1106-preview was used in those cases where inference was required in addition to the JSON mode functionality, as well as if the context length of the prompt exceeded its 16,385 token limit of gpt-3.5-turbo-1106.

Theorem 1.37 (a) Suppose $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^k$, and α is real. Then $|\mathbf{x}| \geq 0$
 Theorem 1.37 (b): Suppose $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^k$, and α is real. Then $|\mathbf{x}| = 0$ if and only if $\mathbf{x} = \mathbf{0}$;
 Theorem 1.37 (c): Suppose $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^k$, and α is real. Then $|\alpha\mathbf{x}| = |\alpha| |\mathbf{x}|$;
 Theorem 1.37 (d): Suppose $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^k$, and α is real. Then $|\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}| |\mathbf{y}|$
 Theorem 1.37 (e): Suppose $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^k$, and α is real. Then $|\mathbf{x} + \mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|$;

Figure 2. gpt-4-1106-preview transformation of 1.

Our prompt consisted of 6 zero shot examples manually drafted, as well as a JSON schema specifying the structure of the output to be produced. In total, it reached just over 10,000 tokens. For this reason, the text was iterated in the smallest gaps possible between subsequent theorems.

As can be seen from 2, gpt-4-1106-preview was able to separate statements and make each self-contained. This was especially important so that when a specific sub-case of such a theorem would be referenced in a future proof, only the necessary context, and not more, would be provided. See [12].

The results of the first step our pre-processing pipeline showcasing this can be found [13]. For each of the three books [4–6], we have separate CSV files for theorems, definitions, corollaries, and propositions, with corresponding labels, statement, and proofs for each.

Currently, we are now in the processes of an using additional script involving gpt-4-1106-preview and gpt-3.5-turbo-1106 in [14] in order to extract from each mathematical statement’s proof all the relevant corresponding statements. In the subsequent section we discuss our current approach so as to have some tangible training data to begin fine tuning our model.

In addition, due to the Mathpix API failing to properly account for the parenthesized formula number delimiters of formulas spread throughout the text, we are building one more script to parse these as efficiently as possible. The code can be found [15]. This is to be able to insert their meaning whenever they arise in proofs.

$$(22) \quad |f_n(t) - A_n| \leq \frac{\epsilon}{3}$$

if $t \in \mathcal{V} \cap E, t \neq x$.
 Substituting the inequalities (20) to (22) into (19), we see that

$$|f(t) - A| \leq \epsilon,$$

provided $t \in \mathcal{V} \cap E, t \neq x$. This is equivalent to (16).

Figure 3. An example from [4] showcasing the necessity of accurately capturing formula labels.

As can be seen from 3, accurately labeling and identifying these formulas are paramount so as to be able to properly insert their meaning when providing our zero-shot context hints via our Lemma Finder.

3.2. Lemma Finder

3.2.1 Training Set

The Lemma Finder is given as input a theorem, and tasked with finding any relevant lemmas, hints, definitions, or other theorems that could be relevant in deriving a solution.

In order to get the training data for the lemma finder, for each theorem that we digitized, we extract from its proof each reference to another theorem, definition and corollary, and retrieve their statements. Using this response per theorem proof, we create a sequence of input theorems and target lemmas such that we have our training dataset. This was done using regex and pandas DataFrame manipulation. See [16].

3.2.2 Fine-tuning job

The final dataset for training is a CSV file with the columns "Theorem" (contains the theorem number), "Theorem Statement", and "References". Using this dataset, we create a JSON Lines file that is required for fine-tuning gpt-3.5-turbo. This file will be the final training set that is supplied for the fine-tuning job. The dataset follows a conversational format and essentially contains following prompting:

```
{ "role": "system", "content": "You are a mathematical analysis tutor. Once given a real analysis statement or question, you provide hints in the form of real analysis statements that are needed to help solve or prove it. },
```

```
{ "role": "user", "content": "Can you provide hints for solving (or proving) the the following question (or statement): Real Analysis theorem statement or question },
```

```
{ "role": "assistant", "content": "(Hint, use the following: theorems needed to solve the question or prove the statement) }
```

Since the model is fine tuned on examples of mathematical statements and the statements that are then used in deriving their proofs, the expected output from our model should be a list of real analysis theorems, definitions, corollaries, and/or propositions that might help in solving or proving said question or statement. This response will serve as useful context for gpt-4, which will use added context and direction to be able to better solve and prove the mathematical analysis questions and statements from the Harvard Math Entrance exam.

3.2.3 Using GPT Assistants

On November 6th, OpenAI introduced the ability to create custom GPTs for specific purposes, called Assistants. An

Assistant can leverage models, tools and knowledge to respond to queries. It has support for Code Interpreter, Retrieval and Function Calling tools.

We created a GPT using the Assistants API. We selected gpt-4-1106-preview for the model, enabled the code interpreter and retrieval tools, and fed it our three books. We provided the following as instructions for the assistant:

"You are a Math Tutor. You are given a set of questions or statements on real, functional and complex analysis, and you respond with the solutions or proofs to each. The training data is a set of books on those topics. They contain theorems, definitions and corollaries and their proofs if applicable. These theorems should help to solve new questions on real, complex and functional analysis."

4. Evaluation Criteria

4.1. Baseline Models

We focused solely on real analysis, functional and complex analysis questions. The initial step involved the conversion of the Harvard math qualifying exams and their corresponding solutions, spanning the years 2012 to 2022, from PDF to \LaTeX format. This conversion was executed using same procedure as in section 2 to ensure consistency across datasets.

We gathered a collection of thirty questions in total. These questions inherently contained explicit instructions; therefore, they were inputted directly into GPT-4 model as given without any additional directives.

The culmination of this process yielded a CSV file [17] which serves as the testing set for our study.

We also used assistants on OpenAI's developer platform to generate some answers, serving as another baseline model. The assistant can independently analyze reference books in pdf format and provide comparisons of the effects using our Lemma Finder, along with the assistant that operate on the same model (GPT-4-1106-preview).

We asked our custom GPT assistant to answer real analysis questions from the Fall 2023 set of Harvard Math PhD Qualifying Exam [18]. This \LaTeX file [19] shows the results.

5. Results

5.1. Questions used

There were three questions in total from the Fall 2023 Math Harvard Entrance Exam that were denoted as Real Analysis in topic.

The first question the was centered on Fourier Series and establishing an inequality using certain boundary conditions of a continuously differentiable function, the second question tested knowledge of compact operators on Hilbert spaces, and the third question required knowledge about in-

equalities in L^p spaces and guarantees on point-wise convergence.

We leave all three of the questions and all answers from each model in the Appendix.

5.2. Evaluation

Table 1. Summary of Question Solving with GPT Versions

Question	Hints Provided	Correctly Solved
GPT-4	No	1, 2
GPT-4	Yes	1, 2, Unsure
Playground GPT	No	1, 2, Unsure

For Professor Drori: I did not want to give a conclusive judgement on the proof when I was unsure myself of the validity of it. So, the rest of the outline will simply comment on the hints provided for Question 3, and that it made its response more rigorous and detailed. Specifically, I am unsure if it is necessary to show that the set on which the function f does not converge approaches 0 as was done in the given solution. If so, then none of them got it correct.

2. (RA) Suppose $\{f_n\}$ are measurable functions on the unit interval $[0, 1] \subset \mathbb{R}$ such that $\{f_n\}$ converges to 0 almost everywhere, and for some $p \geq 1$, we have $\sup_n \|f_n\|_{L^p([0,1])} < \infty$. Show that $\{f_n\}$ converges to 0 in $L^q([0,1])$ for all $1 \leq q < p$.

Figure 4. Third Real Analysis Question

On this third question that we tested, GPT-4 and GPT-4-turbo both followed a similar response in trying to use the Dominated Convergence Theorem to prove the result. (See Appendix C.1, C.2, C.3).

GPT-4 (without hints) specifically was the most simplistic in its response, quoting only the Dominated Convergence Theorem without any explicit derivation relating the q -norms less than p to conclude that they converge to 0 in the limit. (Appendix C.1).

GPT-4-turbo does go the extra step of using Hölder’s inequality to relate the p -norm limsup given for the sequence of functions to those q -norms for $1 \leq q < p$. (Appendix C.3).

And GPT-4 with hints uses both of the above, as well as explicitly mention the problem statement; allowing it to utilize the fact that the norms are uniformly convergent. (Appendix C.2).

6. Discussion

6.1. Lemma Finder

In analyzing the Lemma Finder’s suggested hints, we found that for the first two problems, they proved unuseful to GPT4’s proof. (Appendix A.2, B.2).

Namely, for the second question, we even find in GPT-4’s output that it suggests the provided hints are beyond the scope of the question being asked. (Appendix C.2).

As can be seen in the Appendix, for all problems, the Lemma Finder provides mathematical statements that are more aligned for complex analysis or functional analysis questions. We believe this to be primarily due to the vastly outweighed number of training samples from Complex Analysis and Functional questions [5, 6], as to Real Analysis questions [4].

While the background reading material for the Real Analysis portion of this exam specified these areas as reference, for a large language model like GPT-3.5 that is already likely equipped with this general background, it might be more appropriate to simply fine-tune on the Real Analysis [9] portions alone to specialize for such questions.

6.2. Hallucination

From a preliminary analysis of the hints provided by our Lemma Finder, we were unable to find any hallucinated mathematical statements. So while the the training set imbalance may have lead to unoptimized knowledge domain for this exam, we conjecture that this Fine Tuned model would hallucinate less with regards to mathematical reasoning; although further evaluation would be needed. (Appendix A.2.1, B.2.1, C.2.1).

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A. Results for Question 1

Question

Show that if $f : [0, 1] \rightarrow \mathbb{R}$ is continuously differentiable with $f(0) = f(1) = 0$ then

$$\int_0^1 f'(x)^2 dx \geq \pi^2 \int_0^1 f(x)^2 dx$$

with equality if and only if f is a multiple of $x \mapsto \sin(\pi x)$.

A.1. GPT4 with no hints

To solve this problem, we will use the method of Fourier series and properties of integrals. The key steps are as follows:

1. **Fourier Sine Series**: Since f is continuously differentiable on $[0, 1]$ with $f(0) = f(1) = 0$, it can be represented as a Fourier sine series. Let's denote the Fourier sine

series of f as

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)$$

where b_n are the Fourier coefficients given by

$$b_n = 2 \int_0^1 f(x) \sin(n\pi x) dx.$$

2. **Parseval's Theorem**: Parseval's theorem states that for a function g represented by a Fourier series, the integral of g^2 over its period is equal to the sum of the squares of its Fourier coefficients. Applying this to f gives

$$\int_0^1 f(x)^2 dx = \sum_{n=1}^{\infty} b_n^2.$$

3. **Differentiation**: Differentiating the Fourier series term by term, we get

$$f'(x) = \sum_{n=1}^{\infty} b_n n\pi \cos(n\pi x).$$

Using Parseval's theorem again for f' ,

$$\int_0^1 f'(x)^2 dx = \sum_{n=1}^{\infty} (b_n n\pi)^2.$$

4. **Comparing the Integrals**: Comparing the two integrals, we get

$$\int_0^1 f'(x)^2 dx = \sum_{n=1}^{\infty} (b_n n\pi)^2 \geq \pi^2 \sum_{n=1}^{\infty} b_n^2 = \pi^2 \int_0^1 f(x)^2 dx$$

since $n \geq 1$ for all terms in the series.

5. **Equality Condition**: Equality holds if and only if all terms with $n > 1$ in the series for f are zero, i.e., f is a multiple of $\sin(\pi x)$, since this is the only term where $n = 1$.

Thus, we have shown that

$$\int_0^1 f'(x)^2 dx \geq \pi^2 \int_0^1 f(x)^2 dx,$$

with equality if and only if $f(x)$ is a multiple of $\sin(\pi x)$.

A.2. GPT4 with Hints from GPT3.5

A.2.1 Hints

If a rectangular complex function is of bounded variation in the (x, y) -plane in each of the rectangles

$$R_{h,k} = \{x + iy : 0 \leq x \leq h, 0 \leq y \leq k\}$$

and if h_R and k_R are the one-dimensional projections of $R = R_{h,k}$, then R is of bounded variation in the plane, with

$$V(R) \leq 8V(h_R)V(k_R)$$

In particular, every closed curve has Jordan measure 0.; every simple closed curve, represented parametrically, has winding number 1.; If f is a continuous complex function in the closed unit disc and if $f(z) \neq 0$ at every point in the unit circle $|z| = 1$, then

$$f(0) = [\Pi(f)/(2i\pi)] \log f(1)$$

; If f is a meromorphic function in an annulus $r < |z - a| < R$, and if neither a nor any other point is a pole of f , then the number of zeros, counted with multiplicity, is equal to the number of poles, counted with multiplicity. Theorem 1 If T is a closed (not necessarily bounded) set in the plane, and if $\delta > 0$, then there is a finite or infinite sequence $\{R_k\}$ of rectangles such that a function f in an open set $\Omega \subset X$ (where X is either Euclidean space or the extended plane), which is continuous in Ω except possibly at a set E of Jordan content zero, is said to be Lebesgue integrable over any set $A \subset \Omega$ whose closure is also contained in Ω .

A.3. Output from GPT4

To solve the given problem using the provided hints, we need to adapt these concepts to the context of a real-valued function defined on a real interval. The provided hints relate to complex functions, bounded variation, Jordan measure, winding numbers, and properties of meromorphic functions. These are advanced concepts from complex analysis and measure theory, which are not directly applicable to the simpler setting of a real-valued function on an interval. However, we can draw some inspiration from these concepts to tackle the problem.

The problem is to show:

$$\int_0^1 f'(x)^2 dx \geq \pi^2 \int_0^1 f(x)^2 dx$$

with equality if and only if f is a multiple of $x \mapsto \sin(\pi x)$, for a continuously differentiable function $f : [0, 1] \rightarrow \mathbb{R}$ with $f(0) = f(1) = 0$.

Given that the provided hints are not directly applicable, we can approach the problem using techniques from real analysis:

1. **Wirtinger's Inequality**: This problem is a classic application of Wirtinger's inequality, which states that for a function g defined on $[0, 1]$ that is continuously differentiable and satisfies $g(0) = g(1) = 0$, the following inequality holds:

$$\int_0^1 g'(x)^2 dx \geq \pi^2 \int_0^1 g(x)^2 dx.$$

The equality holds if and only if $g(x)$ is a multiple of $\sin(\pi x)$.

2. **Applying Wirtinger's Inequality**: Applying this inequality to our function f , we directly obtain the desired result. The conditions on f (continuously differentiable and $f(0) = f(1) = 0$) match the conditions required for Wirtinger's inequality.

The specifics of Wirtinger's inequality involve Fourier analysis and properties of sine functions, which are somewhat related to the complex analysis concepts in the hints but are more directly applicable to the problem at hand. Since Wirtinger's inequality is a well-established result, its detailed proof involves somewhat advanced mathematical techniques that go beyond the scope of a simple explanation but can be found in textbooks on real analysis or Fourier analysis.

A.4. Custom GPT Assistant

To show the required inequality and the conditions for equality, we can employ the theory of Sturm-Liouville operators and use integration by parts. Here is a formal proof:

Consider $(f : [0, 1] \rightarrow \mathbb{R})$ which is continuously differentiable with $(f(0) = f(1) = 0)$. Define (L) to be the differential operator $(L[g] = -g'')$, and consider the eigenvalue problem

$$L[g] = \lambda g, \quad g(0) = g(1) = 0.$$

The eigenfunctions of this problem are $(\sin(n\pi x))$ with corresponding eigenvalues $(\lambda_n = (n\pi)^2)$, $(n = 1, 2, 3, \dots)$.

The function (f) can be expanded into a Fourier sine series due to its boundary conditions:

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x),$$

where $(a_n = 2 \int_0^1 f(x) \sin(n\pi x) dx)$. Then we can also write

$$f'(x) = \sum_{n=1}^{\infty} a_n n\pi \cos(n\pi x).$$

Integrating the square of $(f'(x))$ and using Parseval's identity, we have:

$$\int_0^1 f'(x)^2 dx = \sum_{n=1}^{\infty} a_n^2 (n\pi)^2.$$

Similarly, for $(f(x))$, we have:

$$\int_0^1 f(x)^2 dx = \sum_{n=1}^{\infty} a_n^2.$$

Now we can compare these two series term by term. For each $(n \geq 1)$,

$$(n\pi)^2 \geq \pi^2$$

), thus we have:

$$\int_0^1 f'(x)^2 dx = \sum_{n=1}^{\infty} a_n^2 (n\pi)^2 \geq \pi^2 \sum_{n=1}^{\infty} a_n^2 = \pi^2 \int_0^1 f(x)^2 dx.$$

Equality occurs if and only if all terms for $(n \geq 2)$ are zero, that is, $(a_n = 0)$ for all $(n \geq 2)$. This means that $(f(x))$ can only contain the term $(a_1 \sin(\pi x))$, making (f) a multiple of $(x \mapsto \sin(\pi x))$.

Hence, we have shown the required inequality and the condition for equality.

A.5. Solution by Harvard

Syllabus topics: Fourier series

Expand f in a Fourier sine series

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x)$$

Then by Parseval

$$\int_0^1 (f(x))^2 dx = \frac{1}{2} \sum_{n=1}^{\infty} a_n^2$$

For the other integral, we have $f'(x) = \pi \sum_{n=1}^{\infty} n a_n \cos(n\pi x)$ (converging at least in L^2) so, again by Parseval,

$$\int_0^1 (f'(x))^2 dx = \frac{\pi^2}{2} \sum_{n=1}^{\infty} n^2 a_n^2$$

The desired inequality follows because $n^2 a_n^2 \geq a_n^2$ with equality if and only if $n = 1$ or $a_n = 0$.

Remarks: This is Wirtinger's inequality, whose applications include a proof of the isoperimetric inequality for plane curves. The Fourier sine expansion is a natural tool because the ratio $\int_0^1 (f'(x))^2 dx / \int_0^1 (f(x))^2 dx$ is a Rayleigh quotient for the operator $D : f \mapsto f'$, as the Fourier sine series expands f with respect to an orthogonal eigenbasis for D . There is also an elementary but trickier proof by integration by parts, starting by writing $f(x) = g(x) \sin(\pi x)$ for some continuous $g : [0, 1] \rightarrow \mathbb{R}$ that is differentiable on $(0, 1)$, and eventually proving

$$\int_0^1 (g(x) \sin(\pi x))^2 dx \geq \int_0^1 (g(x) \sin(\pi x))' ^2 dx$$

with equality if and only if $g'(x) = 0$ for all x .

B. Results for Question 2

Additional appendix content.

Question

A bounded operator $T : \mathcal{H} \rightarrow \mathcal{H}$ on a Hilbert space \mathcal{H} is said to be compact if for any sequence $\{x_n\}_{n=1}^{\infty}$ in \mathcal{H} such that $\sup_n \|x_n\| < \infty$, the sequence $\{Tx_n\}_{n=1}^{\infty}$ contains a convergent subsequence.

Suppose that $T : \mathcal{H} \rightarrow \mathcal{H}$ is a compact operator. Show that for any $\sigma \in \mathbb{C} \setminus \{0\}$, the eigenspace $\{\psi \in \mathcal{H} : T\psi = \sigma\psi\}$ is finite-dimensional.

Show that if $T : \mathcal{H} \rightarrow \mathcal{H}$ is compact and $S : \mathcal{H} \rightarrow \mathcal{H}$ is bounded, then $T \circ S$ is compact.

Suppose that $T : \mathcal{H} \rightarrow \mathcal{H}$ is compact and invertible. Show that \mathcal{H} is finite-dimensional.

B.1. GPT4 with no hints

These problems involve the properties of compact operators on Hilbert spaces. Let's solve them step by step:

1. Eigenspace of a Compact Operator is Finite-Dimensional

****Problem Statement**:** If $T : \mathcal{H} \rightarrow \mathcal{H}$ is a compact operator, show that the eigenspace for any $\sigma \in \mathbb{C} \setminus \{0\}$, $\{\psi \in \mathcal{H} : T\psi = \sigma\psi\}$, is finite-dimensional.

****Solution**:** 1. ****Contradiction Assumption**:** Assume the eigenspace is infinite-dimensional. We can find a sequence of orthonormal vectors $\{\psi_n\}$ in this eigenspace.

2. ****Compactness of T **:** Since T is compact, $\{T\psi_n\}$ must have a convergent subsequence. 3. ****Eigenvector Property**:** $T\psi_n = \sigma\psi_n$ for each n , so $\{T\psi_n\} = \{\sigma\psi_n\}$.

4. ****Orthogonality Contradiction**:** The orthonormality of $\{\psi_n\}$ means they cannot have a convergent subsequence (except trivially to zero, which contradicts $\sigma \neq 0$).

5. ****Conclusion**:** Hence, the eigenspace must be finite-dimensional.

2. Compactness of $T \circ S$

****Problem Statement**:** Show that if $T : \mathcal{H} \rightarrow \mathcal{H}$ is compact and $S : \mathcal{H} \rightarrow \mathcal{H}$ is bounded, then $T \circ S$ is compact.

****Solution**:** 1. ****Sequence Consideration**:** Take any bounded sequence $\{x_n\}$ in \mathcal{H} . 2. ****Boundedness of S **:** $\{Sx_n\}$ is also a bounded sequence since S is bounded.

3. ****Compactness of T **:** The compactness of T implies $\{TSx_n\} = \{T(Sx_n)\}$ has a convergent subsequence. 4. ****Conclusion**:** Therefore, $T \circ S$ is compact.

3. Compact and Invertible Implies Finite-Dimensional

****Problem Statement**:** If $T : \mathcal{H} \rightarrow \mathcal{H}$ is compact and invertible, show that \mathcal{H} is finite-dimensional.

****Solution**:** 1. ****Contradiction Assumption**:** Assume \mathcal{H} is infinite-dimensional. 2. ****Sequence Construction**:** Construct an orthonormal sequence $\{x_n\}$ in \mathcal{H} . 3. ****Compactness of T **:** $\{Tx_n\}$ must have a convergent subsequence. 4. ****Invertibility of T **:** $Tx_n = 0$ implies $x_n = 0$, but this contradicts the orthonormality of $\{x_n\}$.

5. ****Contradiction****: This leads to a contradiction, so \mathcal{H} must be finite-dimensional.

These proofs involve fundamental concepts in functional analysis, particularly the properties of compact operators on Hilbert spaces. Each solution is presented in a concise form, focusing on the key steps necessary for the proofs.

B.2. GPT4 with hints from fine-tuned GPT3.5

B.2.1 Hints

As in the case of bounded linear operators on Banach spaces, one can show that

T is compact if and only if T maps bounded sets to precompact sets.

More explicitly, $T : \mathcal{X} \rightarrow \mathcal{Y}$, where \mathcal{X} is a Frchet space and \mathcal{Y} is a topological vector

space), is compact if and only if for every bounded subset M of \mathcal{X} , TM (which is easily seen

to be bounded) has compact closure in \mathcal{Y} .; If \mathcal{J} is a compact subset of the closed right half-plane, then the sets

$M_R(\sigma)$ and $M_B(\sigma)$ are sequentially compact subsets of C_R .

For $1 < p < \infty$ B_M is the closed unit ball of M , and for $p = \infty$ its closure M_c is the containing unit ball which has the additional property of being sequentially compact.

Suppose $A_n \in L(sp)_m, 1 \leq p \leq \infty, \Lambda \in L(sp)_\mu$, and

$$\Lambda A_n = A_n \Lambda \quad (n = 1, 2, 3, \dots)$$

Then λ [which was defined just before (5.15)], is a bounded homomorphism of $L(X_n)$, and

$$\sigma(\Lambda, A) = \sigma(\lambda) \quad \text{and} \quad \rho(\Lambda, A) = \rho(\lambda)$$

Suppose $A_n \in L(sp)_m, 1 \leq p \leq \infty, \Lambda \in L(sp)_\mu$, and

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$$\sigma(\Lambda, A) = \sigma(\lambda) \quad \text{and} \quad \rho(\Lambda, A) = \rho(\lambda)$$

A trace ρ is a regular countably additive complex set function that is defined on the collection of all subsets in the

associated ring \mathfrak{A} , which are the finite unions of sets of the form $E(\lambda)$. For every set $E \in \mathfrak{A}$ and for every sequence $E_n \in \mathfrak{A} (n = 1, 2, 3, \dots)$, the condition

$$E = \bigcup_{n=1}^{\infty} (E_n \setminus E_{n+1})$$

implies

$$\rho(E) = \lim_{n \rightarrow \infty} \rho(E_n)$$

B.2.2 GPT4 output with hints

To solve this question, we need to address three separate but related statements about compact operators on a Hilbert space \mathcal{H} . Let's address each statement one by one:

1. ****Eigenspace of a Compact Operator is Finite-Dimensional****:

We are given a compact operator $T : \mathcal{H} \rightarrow \mathcal{H}$ and need to show that for any $\sigma \in \mathbb{C} \setminus \{0\}$, the eigenspace $\{\psi \in \mathcal{H} : T\psi = \sigma\psi\}$ is finite-dimensional.

To prove this, assume for contradiction that the eigenspace corresponding to σ is infinite-dimensional. Then there exists an infinite sequence of orthonormal vectors $\{\psi_n\}$ in this eigenspace (by the property of Hilbert spaces). Since T is compact, $\{T\psi_n\}$ should have a convergent subsequence. However, $T\psi_n = \sigma\psi_n$, and since $\{\psi_n\}$ is orthonormal, it does not have a convergent subsequence (no subsequence can converge because each pair of elements is orthogonal and has norm 1). This contradiction implies that our assumption is false, and thus the eigenspace must be finite-dimensional.

2. ****Composition of a Compact and a Bounded Operator is Compact****:

Let $T : \mathcal{H} \rightarrow \mathcal{H}$ be a compact operator and $S : \mathcal{H} \rightarrow \mathcal{H}$ be a bounded operator. We need to show that $T \circ S$ is compact.

Take any bounded sequence $\{x_n\}$ in \mathcal{H} . Since S is bounded, $\{Sx_n\}$ is also a bounded sequence. As T is compact, $\{TSx_n\} = \{T(Sx_n)\}$ has a convergent subsequence. Therefore, $T \circ S$ maps bounded sequences to sequences with convergent subsequences, which means $T \circ S$ is compact.

3. ****Compact and Invertible Operator Implies Finite-Dimensional Space****:

Suppose $T : \mathcal{H} \rightarrow \mathcal{H}$ is compact and invertible. We need to show that \mathcal{H} is finite-dimensional.

Since T is invertible, its kernel is trivial. This means 0 is not an eigenvalue of T . By the spectral theorem for compact operators, the non-zero spectrum of T consists of eigenvalues with finite multiplicity, and they accumulate only at 0. However, since 0 is not an eigenvalue of T , T has only a finite number of eigenvalues. Thus, \mathcal{H} can be decomposed into a direct sum of finite-dimensional eigenspaces of T , implying \mathcal{H} itself must be finite-dimensional.

This addresses each of the three statements in your question.

B.3. Custom GPT Assistant

All the operators considered are linear.

Question 1: Show that for any nonzero complex number σ , the eigenspace $\{\psi \in \mathcal{H} : T\psi = \sigma\psi\}$ corresponding to a compact operator $T : \mathcal{H} \rightarrow \mathcal{H}$ is finite-dimensional.

Proof: Suppose by contradiction that the eigenspace $E_\sigma = \{\psi \in \mathcal{H} : T\psi = \sigma\psi\}$ is infinite-dimensional for some nonzero $\sigma \in \mathbb{C}$. Then there exists an infinite sequence of orthonormal vectors $\{\psi_n\}_{n=1}^\infty$ in E_σ . Since T is compact, the sequence $\{T\psi_n\}_{n=1}^\infty$ must have a convergent subsequence in \mathcal{H} . However, for $m \neq n$, $\|T\psi_n - T\psi_m\| = |\sigma|\|\psi_n - \psi_m\| = 2|\sigma|$, which contradicts the sequence being Cauchy and thus converging. Therefore, E_σ must be finite-dimensional. \square

Question 2: Show that if $T : \mathcal{H} \rightarrow \mathcal{H}$ is compact and $S : \mathcal{H} \rightarrow \mathcal{H}$ is bounded, then $T \circ S$ is compact.

Proof: Let $\{x_n\}_{n=1}^\infty$ be a bounded sequence in \mathcal{H} . Since S is bounded, the sequence $\{Sx_n\}_{n=1}^\infty$ is also bounded. Therefore, by the compactness of T , the sequence $\{T(Sx_n)\}_{n=1}^\infty$ has a convergent subsequence, showing that $T \circ S$ is compact. \square

Question 3: Suppose that $T : \mathcal{H} \rightarrow \mathcal{H}$ is compact and invertible. Show that \mathcal{H} is finite-dimensional.

Proof: Assume for the sake of contradiction that \mathcal{H} is infinite-dimensional. Then it is possible to construct an orthonormal sequence $\{x_n\}_{n=1}^\infty$ in \mathcal{H} . Since T is invertible, none of the Tx_n can be zero, and $\|Tx_n - Tx_m\| = \|T(x_n - x_m)\|$ implies that the sequence $\{Tx_n\}$ does not have any convergent subsequences, as $x_n \perp x_m$ for $m \neq n$. This contradicts T being compact, hence \mathcal{H} must be finite-dimensional. \square

B.4. Solution

Syllabus topics: Fourier series

Solution Expand f in a Fourier sine series

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x)$$

Then by Parseval

$$\int_0^1 (f(x))^2 dx = \frac{1}{2} \sum_{n=1}^{\infty} a_n^2$$

For the other integral, we have $f'(x) = \pi \sum_{n=1}^{\infty} n a_n \cos(n\pi x)$ (converging at least in L^2) so, again by Parseval,

$$\int_0^1 (f'(x))^2 dx = \frac{\pi^2}{2} \sum_{n=1}^{\infty} n^2 a_n^2$$

The desired inequality follows because $n^2 a_n^2 \geq a_n^2$ with equality if and only if $n = 1$ or $a_n = 0$.

Remarks: This is Wirtinger's inequality, whose applications include a proof of the isoperimetric inequality for plane curves. The Fourier sine expansion is a natural tool because the ratio $\int_0^1 (f'(x))^2 dx / \int_0^1 (f(x))^2 dx$ is a Rayleigh quotient for the operator $D : f \mapsto f'$, as the Fourier sine series expands f with respect to an orthogonal eigenbasis for D . There is also an elementary but trickier proof by integration by parts, starting by writing $f(x) = g(x) \sin(\pi x)$ for some continuous $g : [0, 1] \rightarrow \mathbb{R}$ that is differentiable on $(0, 1)$, and eventually proving

$$\int_0^1 (g(x) \sin(\pi x))^2 dx \geq \int_0^1 (g(x) \sin(\pi x))' dx$$

with equality if and only if $g'(x) = 0$ for all x .

C. Results for Question 3

Question

Suppose $\{f_n\}$ are measurable functions on the unit interval $[0, 1] \subset \mathbb{R}$ such that $\{f_n\}$ converges to 0 almost everywhere, and for some $p \geq 1$, we have $\sup_n \|f_n\|_{L^p([0,1])} < \infty$. Show that $\{f_n\}$ converges to 0 in $L^q([0,1])$ for all $1 \leq q < p$.

C.1. GPT4 with no hints

To solve this problem, we'll use the concept of convergence in L^p spaces and the properties of measurable functions. We are given that:

- $\{f_n\}$ are measurable functions on the unit interval $[0, 1]$.
- $\{f_n\}$ converges to 0 almost everywhere.
- For some $p \geq 1$, $\sup_n \|f_n\|_{L^p([0,1])} < \infty$.

We need to show that $\{f_n\}$ converges to 0 in $L^q([0, 1])$ for all $1 \leq q < p$.

****Solution:****

We use the Dominated Convergence Theorem (DCT). For the DCT, we need two conditions to be satisfied: 1. The sequence of functions $\{f_n\}$ converges pointwise almost everywhere to a function f (which in this case is the zero function). 2. There exists an integrable function g such that $|f_n(x)| \leq g(x)$ almost everywhere for all n .

The first condition is satisfied by assumption, as $\{f_n\}$ converges to 0 almost everywhere.

For the second condition, we need to find such a function g . Since $\sup_n \|f_n\|_{L^p([0,1])} < \infty$, there exists an $M < \infty$ such that $\|f_n\|_{L^p} \leq M$ for all n . This implies that $|f_n|^p \leq M^p$ in the L^1 sense, as the L^p norm is the p -th root of the integral of $|f_n|^p$.

Now, define $g(x) = M^p$ for all $x \in [0, 1]$. Clearly, $|f_n(x)|^p \leq g(x)$ almost everywhere, and since $g(x)$ is constant and finite over $[0, 1]$, it is integrable.

With both conditions of the DCT satisfied, we can conclude that $\int_0^1 |f_n|^q \rightarrow \int_0^1 0^q = 0$ as $n \rightarrow \infty$ for $1 \leq q < p$. This implies that $\|f_n\|_{L^q} \rightarrow 0$ as $n \rightarrow \infty$ for $1 \leq q < p$, which is what we wanted to prove.

C.2. GPT4 with hints from fine-tuned GPT3.5

C.2.1 Hints

If $\mu(X) < \infty$ and $p > 1$, then $C_c(X)$ is dense in $L^p(\mu)$. Suppose X is locally compact, there is a positive Borel measure λ on X with the following three properties: (a) $0 < \lambda(K) < \infty$ if the compact set K satisfies HYP on X . (b) If E and α isoscrews GIF 2.2.1 *[0, infy) then $\lambda^*(E) = \alpha(0)\lambda^*(X)$ implies that $\lambda^*(E) = \alpha t \lambda^*(X)$ for every $t \in [0, \infty)$. (c) If E is as in (b) and if m is an integer then $\lambda^*(E + m) = \lambda^*(E)$. If, in addition, X is separable, then there is a complete, separable metric in F -algebra M of all real-valued Borel measurable functions on X , which is unique in the sense that every Borel measurable function whose domain is a Borel set in X is Leb_X -equivalent to som;The function $p:=\tilde{\iota}[0,]=\tilde{\iota}[\cdot]$, a resolution of R . whose properties (theorems 2.1.4 to 2.1.6) are stated typographically. Note by MCSWeb

$$p(E) = \int_{\Omega} p_E d\mu, \quad \text{for every } E \text{ Borel measurable in } \Omega.$$

$$(*) \quad \lambda = p \circ \mu \quad \text{on } B_{\Omega}$$

The Borel sets E that satisfy p_E holomorphic in $\tilde{\iota}X_{\tilde{\iota}}$, and a large class of such sets is the class H . A function $f : \Omega \rightarrow V$ is said to be holomorphic in Ω , written $f \in H(\Omega)$, if f is continuous at each point of Ω , and if the following limiting-differene relations holds at each point and for all h in the given

$$\lim_{h \rightarrow 0} \left(\frac{D^j}{dh^j} f \right) (z) = f^{(j)}(z) \quad (j = 0, 1, 2, \dots)$$

, D^j standing for the usual multiple derivative and the limit being taken within Ω .Note by MCSWeb:= $\tilde{\iota}\tilde{\iota}$,= $\tilde{\iota}\tilde{\iota}$

$$(W \cdot \tilde{W} f)(x) = \int_{\Omega} W(x, \xi) \tilde{W}(x, \xi) f(\xi) d\mu(\xi), \quad x \in \Omega, \quad (\text{ii}')$$

every $x \in \Omega$, both sides of $\tilde{\iota}\tilde{\iota}$, are the identity operator (i.e., on $L^2(\mu)$), and, therefore, so are the two sides of $\tilde{\iota}\tilde{\iota}$, $\tilde{\iota}\tilde{\iota}\tilde{\iota}\tilde{\iota}(f)$, for every $f \in L^2(\mu)$, ;Suppose that $h \in H(D)$ in the open unit disk.

$$\ll N(\lambda) = \{f \in A : pf \in \lambda\} \gg$$

;The first part of the proof reads: By the maximum principle,

$\text{Re } h_{ext} \left(\frac{\lambda+z}{\lambda-z} \right) \leq 0$ if $|z| < 1$, therefore also $\text{Re } h_{ext} \left(\frac{1+z}{1-z} \right) \leq 0$ and hence

$$h_{ext} \leq h_{ext}(1/2, t)(t) = h_{ext}(t)$$

on $[0, 2] Q$.

Then we have

$$Lf = (I + R_f) f \quad (f \in H^{\infty}(D)), \text{Leb}_D \text{-a.e.}$$

$$(*) \quad \lambda = p \circ \mu, \quad \text{on } B_{\Omega}$$

$[H^{\infty}(D)d\mu]$ is a closed subspace of $L^{\infty}(D, d\mu)$, ;Suppose that $h \in H(D)$ in the open unit disk.;Suppose that $h \in H(D)$ in the open unit disk.;Suppose that $h \in H(D)$ in the open unit disk. $\mu \{t : h_{ext}(t) = 0\} = 0$ (Rudin-FA[1987,ex.18.3.5]), ;Suppose that $h \in H(D)$ in the open unit disk. $\mu \{t : h_{ext}(t) = 0\} = 0$ (Rudin-FA[1987,ex.18.3.5]), Binet's or catalecticant's formula: If γ is an arc with endpoints w_1 and w_2 , and if α and β are numbers, then each of the mixed second order derivatives $\tilde{\iota}\tilde{\iota}[\alpha, \beta]\psi(z) = \lambda^2\psi(z) = 0$, $[\beta, \alpha]\psi(z) = \lambda^2\psi(z) = 0$, $[\alpha, \alpha]\psi(z) = \lambda\psi(z) = 0$, $[\beta, \beta]\psi(z) = \lambda\psi(z) = 0$.

Suppose

$$\begin{cases} Ly = f, & \text{(a)} \\ \log \rho(y_0) = \int_0^{2\pi} \frac{d\theta}{2\pi} P(ei\theta), & \text{(b)} \end{cases}$$

$$C(\rho(y_0)) \in H_{\rho(y_0)} \quad \text{and} \quad R_{\tau} \circ C = C \text{ on } \hat{\Omega}$$

$$\Phi(\tau) = \int_{\Gamma} C(z) R_{\tau} R_{\tilde{z}ez} y_0 ds(z), \quad \tau \in \Omega.$$

$$\frac{d}{d\tau} \Phi(\tau) = 0 \text{ if } \text{Im } \tau > 0 \text{ and}$$

you obtain an equality of the λ 's in the following commutative diagram:

$$\begin{array}{ccc} H^{\infty}(D) & \xrightarrow{L} & L_{ext}^{\infty}[0, 2] \\ \downarrow p & & \downarrow p_* \\ C & \xrightarrow{L} & L_{ext}^{\infty}[\alpha(0), \alpha(2)] \\ \downarrow & & \downarrow Q_* \\ V & \xrightarrow{Q} & Y \end{array}$$

Note that $Q_* p_* = Q$, since $p_*(gt) = gp_*(t)$ and $Q(g(t)) = Q_1(t)$ by definition of Q , ;[19, 182e]Formulation of the theorems of representation]; $\tilde{\iota}\tilde{\iota}$;Suppose that $h \in H(D)$ in the open unit disk.;Suppose that

$h \in H(D)$ in the open unit disk. Suppose that $h \in H(D)$ in the open unit disk. If $|\lambda| = 1$ almost everywhere on T , then λ is a constant multiple of a Blaschke factor.

Note. If $h_{ext} \in L^1(\mu)$ for some pointwise radial measure μ on T and $(h \circ \text{ext})(t) = 0$ for a.e. t , then $(h \circ \text{ext})(t) = 0$ for every t . Such functions h are called cyclic (on D). Wordy but easy. Let $h_{ext} \in L^1(\mu)$ for some pointwise radial measure μ on T and $(h \circ \text{ext})(t) = 0$ for a.e. t , then $(h \circ \text{ext})(t) = 0$ for every t . Such functions h are called cyclic (on D). If the complex power series $\sum c_n z^n$ converges at a point p then the series

$$\sum_{n=0}^{\infty} c_n r^n$$

is a Carleson measure, no (positive) integer powers of z are included in the series (save possibly the term a_0), Poisson's integral for the circle:

$$|P(T)| \leq \int_{\Gamma} |C| ds \leq 2\pi \max_{t \in \Gamma} |C(t)| = 2\pi \kappa \quad (\lambda \in \Omega)$$

$$Q(C) = C \circ e$$

$$(X)(Y + Z) = XY + XZ$$

$$(X + Y)Z = XZ + YZ$$

$$[c(X)]Z = X[CZ] = C(XZ)$$

$\lambda \in B_{\Omega}$; Suppose that $h \in H(D)$ in the open unit disk.

$$(X)(Y + Z) = XY + XZ$$

$$(X + Y)Z = XZ + YZ$$

$$[c(X)]Z = X[CZ] = C(XZ)$$

$\lambda \in B_{\Omega}$; Suppose that V and Y are as postulated, and that G is a Banach space, $\ll l.MCSE[2, 340d] \gg$ Suppose $h \in H(D)$ in the open unit disk, and that every $h(w) = 0$. Then $h \equiv 0$. Feller . ; ;

$$F(z) = G \circ C(z) = G \circ R \circ Q(z) = G \circ R(\tau) = G(\tau)$$

If $f \in A$ and if $f \circ Q_1 = f$ on $[0, 1]$, the range of f contains a neighborhood of $\text{ext} X$. ; Suppose

$$\begin{cases} Ly = f, & (a) \\ \log \rho(y_0) = \int_0^{2\pi} \frac{d\theta}{2\pi} P(e^{i\theta}), & (b) \end{cases}$$

$$\Phi(\tau) = \int_{y_0}^{\tau} \frac{d\rho(\zeta)}{\rho(\zeta)}$$

$$\Phi(\tau) = \int_{y_0}^{\tau} \frac{d\rho(\zeta)}{\rho(\zeta)}; If$$

$f \in A$ and if $|f(Q_1(t))| \geq 1$ for every $t \in [-1, 1]$, then f has no zeros in D . ; Let Suppose

$$TV \ni x \rightarrow J(x) \in X$$

; ; Suppose G is Zolotarev in

$$\Omega = \{z = x + iy : 0 < x < \pi, y > 0\}$$

and that Ω can be mapped by a partial fraction decomposition into the exterior of the closed unit disk. ; If the three conclusions (Lemma 1.4.1 [E] Haar 1.6.6) ;

$$d^2 y_i(t) \frac{d^2 y_i(t)}{dt^2 + \lambda_i^2 y_i(t) = 0} \quad (-\infty < t < \infty)$$

; If [3], $(KJ = J$ on Q , [4, 649b] Theorem C) ;

$$\lambda_i^2 = \int_{X_i} \phi'(\xi) dm \quad (i = 1, 2)$$

; ; Suppose that $h \in H(D)$ in the open unit disk. ; If e_1, e_2, \dots, e_d is the

standard basis vector of C^d , let

$$P(t) = P(e^{it})$$

and

$$X(t) = P(e^{it}) \\ [TQ_1(e^{it})]$$

These remarks complete the proof [Feller 1950]. It is convenient to collect the results as follows. If G is a Stieltjes transform

Suppose

$$\begin{cases} Ly = f, & \text{(a)} \\ \log \rho(y_0) = \int_0^{2\pi} \frac{d\theta}{2\pi} P(e^{i\theta}), & \text{(b)} \end{cases}$$

B and \bar{B} often denote the closed and open balls

$$B_\rho(x_0) = \{x : \rho(x, x_0) < \rho\}, \text{ etc.}$$

Suppose G is Zolotarev in

$$\Omega = \{z = x + iy : 0 < x < \pi, y > 0\}$$

and that Ω can be mapped by a partial fraction decomposition into the exterior of the closed unit disk.. Suppose G is a

harmonic function in the upper half plane and $L = 0$ in D , Suppose G is a harmonic function in the upper half plane and $L \geq 0$ in D , ;

$$h_r = h \circ e^r$$

Suppose G is a harmonic function in the upper half plane and $L \geq 0$ in D ,

Suppose that $h \in H(D)$ in the open unit disk. If the complex power series $\sum c_n z^n$ converges at a p

Also,

$$Q_1(t) = \frac{1-t}{1+t}$$

, and

$$e_{xt}(-t^2) = e^{it} \quad (t \in (0, \pi/2))$$

.Then

$$q - (0, \frac{\pi}{2}) > I,$$

and, by necessity,

$$e_{xt}(t^2) = e^{it}$$

C.2.2 GPT4 output with hints

To solve the given problem, let's break it down into key points and apply the provided information step by step.

1. **Given Information**: - $\{f_n\}$ are measurable functions on the unit interval $[0, 1] \subset \mathbb{R}$. - $\{f_n\}$ converges to 0 almost everywhere. - For some $p \geq 1$, $\sup_n \|f_n\|_{L^p([0,1])} < \infty$. - We need to show that $\{f_n\}$ converges to 0 in $L^q([0, 1])$ for all $1 \leq q < p$.

2. **Approach**: - **Convergence Almost Everywhere**: Since $\{f_n\}$ converges to 0 almost everywhere, for almost every $x \in [0, 1]$, $f_n(x) \rightarrow 0$ as $n \rightarrow \infty$. - **Uniform Boundedness in L^p** : The condition

$\sup_n \|f_n\|_{L^p([0,1])} < \infty$ implies that the sequence $\{f_n\}$ is uniformly bounded in the L^p norm.

3. **Using Density of Continuous Functions**: - Since $C_c(X)$ (continuous functions with compact support) is dense in $L^p(\mu)$ for $p > 1$ and $\mu(X) < \infty$, we can approximate f_n by a sequence of continuous functions g_n in L^p norm.

4. **Convergence in L^q Norm for $1 \leq q < p$** : - By Hölder's inequality, for $1 \leq q < p$, the L^q norm of a function is less than or equal to its L^p norm. This is because $[0, 1]$ is of finite measure. - Since $\{f_n\}$ converges to 0 almost everywhere and is uniformly bounded in L^p , by the Dominated Convergence Theorem, $\{f_n\}$ also converges to 0 in L^q for every $q < p$.

5. **Conclusion**: - Therefore, $\{f_n\}$ converges to 0 in $L^q([0, 1])$ for all $1 \leq q < p$.

This approach utilizes the provided information about the density of continuous functions in L^p spaces, the Dominated Convergence Theorem, and Hölder's inequality to establish the required convergence.

C.3. Custom GPT Assistant

Question: Suppose $\{f_n\}$ are measurable functions on the unit interval $[0, 1] \subset \mathbb{R}$ such that $\{f_n\}$ converges to 0 almost everywhere, and for some $p \geq 1$, we have $\sup_n \|f_n\|_{L^p([0,1])} < \infty$. Show that $\{f_n\}$ converges to 0 in $L^q([0, 1])$ for all $1 \leq q < p$.

Proof: Given the conditions, we apply the Dominated Convergence Theorem for L^q spaces, where $1 \leq q < p$.

To use this theorem, we first show that $|f_n|^q$ is dominated by an integrable function that does not depend on n .

We know $\sup_n \|f_n\|_{L^p}^p = M < \infty$, which means that for any n , $\int_0^1 |f_n|^p dx \leq M$. Now, for any $1 \leq q < p$, we can use Hölder's inequality with exponents $\frac{p}{q}$ and $\frac{p}{p-q}$ to get

$$\left(\int_0^1 |f_n|^q dx \right)^{\frac{p}{q}} \leq \left(\int_0^1 |f_n|^p dx \right) \left(\int_0^1 1^{\frac{p}{p-q}} dx \right)^{\frac{p-q}{q}} = \int_0^1 |f_n|^p dx \leq M.$$

So, $\int_0^1 |f_n|^q dx \leq M^{\frac{q}{p}}$, which implies $\|f_n\|_{L^q}^q \leq M^{\frac{q}{p}}$ for all n .

Next, since the functions f_n converge to 0 almost everywhere and $|f_n|^q$ is bounded above by $M^{\frac{q}{p}}$ which is integrable on $[0, 1]$,

$$\lim_{n \rightarrow \infty} \int_0^1 |f_n|^q dx = 0.$$

This limit follows directly due to the Dominated Convergence Theorem.

Thus, we have shown that $\{f_n\}$ converges to 0 in $L^q([0, 1])$ for all $1 \leq q < p$. \square

C.4. Solution

Syllabus topics: L^p spaces

Solution: Fix any $\varepsilon > 0$. Set $\mathcal{E}_{\varepsilon, n} := \{x \in [0, 1] : |f_n(x)| \geq \varepsilon\}$. We first have

$$\begin{aligned} \int_0^1 |f_n(x)|^q dx &= \int_{\mathcal{E}_{\varepsilon, n}} |f_n(x)|^q dx + \int_{\mathcal{E}_{\varepsilon, n}^c} |f_n(x)|^q dx \\ &\leq \int_{\mathcal{E}_{\varepsilon, n}} |f_n(x)|^q dx + \varepsilon^q. \end{aligned}$$

By Hölder,

$$\int_{\mathcal{E}_{\varepsilon, n}} |f_n(x)|^q dx \leq \left(\int_{\mathcal{E}_{\varepsilon, n}} |f_n(x)|^p dx \right)^{\frac{q}{p}} (\mu(\mathcal{E}_{\varepsilon, n}))^{\alpha(p, q)}$$

By assumption, the first factor on the RHS is bounded uniformly in n and $\mu(\mathcal{E}_{\varepsilon, n}) \rightarrow 0$ as $n \rightarrow \infty$. Thus,

$$\limsup_{n \rightarrow \infty} \int_0^1 |f_n(x)|^q dx \leq \varepsilon^q$$

Since $\varepsilon > 0$ is arbitrary, this means the limsup on the left-hand side is 0.